P173.

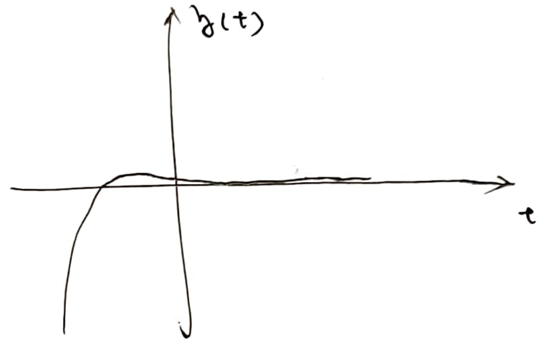
14. Sol:

chara equ: $\lambda^2 + 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = -2$

$\Rightarrow y_1(t) = e^{-2(t+1)} \quad y_2(t) = te^{-2(t+1)}$

$$\begin{cases} C_1 y_1(-1) + C_2 y_2(-1) = y(-1) \\ C_1 y_1'(-1) + C_2 y_2'(-1) = y'(-1) \end{cases}$$

$\Rightarrow \begin{cases} C_1 - C_2 = 2 \\ -2C_1 + 3C_2 = 1 \end{cases}$



$\Rightarrow \begin{cases} C_1 = 7 \\ C_2 = 5 \end{cases}$

i.e. $y(t) = 7e^{-2(t+1)} + 5te^{-2(t+1)}$

$y \to 0$ as $t \to \infty$.

21. Sol:

Obs: $\dfrac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}$ is a linear combination of $e^{r_2 t}$ & $e^{r_1 t}$,

hence $\varphi(t; r_1, r_2)$ is a sol

$$\lim_{r_2 \to r_1} \varphi(t; r_1, r_2) = \lim_{r_2 \to r_1} \frac{e^{r_2 t} - e^{r_1 t}}{r_2 - r_1}$$

$$= \lim_{r_2 \to r_1} \frac{e^{r_1 t}\left(e^{(r_2 - r_1)t} - 1\right)}{r_2 - r_1} = e^{r_1 t} t$$

i.e. $te^{r_1 t}$ is a sol in the case of equal roots.

31. Sol:

set $y(x) = u(x)\, y_1(x)$

then $y'(x) = u'(x)\, y_1(x) + u(x)\, y_1'(x)$

$y''(x) = u''(x)\, y_1(x) + u'(x)\, y_1'(x) + u'(x)\, y_1'(x) + u(x)\, y_1''(x)$.

then $y'' + \delta(xy' + y)$

$= (u'' + \delta x u' + \delta u)\, y_1 + (2u' + \delta x u)\, y_1' + u\, y_1''$

notice $y_1(x) = e^{-\frac{\delta}{2}x^2}$

$\Rightarrow y_1'(x) = -\delta x\, e^{-\frac{\delta}{2}x^2} = -\delta x\, y_1$

$y_1''(x) = (\delta x)^2 e^{-\frac{\delta}{2}x^2} - \delta e^{-\frac{\delta}{2}x^2} = (\delta x)^2 y_1 - \delta y_1$.

then $\left((\delta x)^2 - \delta\right) u - \delta x\,(2u' + \delta x u) + \left(u'' + \delta x\, u' + \delta u\right) = 0$.

i.e. $u'' - \delta x\, u' = 0$

$\Rightarrow \frac{d}{dx}\left(e^{-\frac{\delta}{2}x^2} u'\right) = 0$

$\Rightarrow e^{-\frac{\delta}{2}x^2} u' = C$

$\Rightarrow u = C_1 \int e^{\frac{\delta}{2}x^2}\, dx + C_2$

i.e. $y(x) = C_1 e^{-\frac{\delta}{2}x^2} \int_0^x e^{\frac{\delta}{2}t^2}\, dt + C_2 e^{-\frac{\delta}{2}x^2}$.

37. Sol:

notice that $a\lambda^2 + b\lambda + c = 0$

$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 = -\frac{b}{a} < 0 \\ \lambda_1 \lambda_2 = \frac{c}{a} > 0 \end{cases} \Rightarrow \begin{matrix} \lambda_1 < 0 \\ \lambda_2 < 0 \end{matrix} \Rightarrow \begin{matrix} e^{\lambda_1 t} \to 0 \\ e^{\lambda_2 t} \to 0 \end{matrix}$ as $t \to \infty$.

i.e. $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \to 0$ as $t \to \infty$.

38. Sol:

a> Chara equ becomes $a\lambda^2 + c = 0$, which has two complex roots.

i.e. the sol has the form:

$y(t) = c_1 \cos\alpha t + c_2 \sin\alpha t$, which is bdd for all t.

b> Chara equ: $a\lambda^2 + b\lambda = 0 \Rightarrow \lambda_1 = 0 \quad \lambda_2 = -\frac{b}{a} < 0$

i.e. $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$= c_1 + c_2 e^{-\frac{b}{a}t}$

notice that $e^{-\frac{b}{a}t} \to 0$ as $t \to \infty$,

we know that $y(t) \to c_1$ as $t \to \infty$,

Obs: $y(0) = c_1 + c_2 = y_0$,

$y'(0) = -\frac{b}{a} c_2 = y_0'$

$\Rightarrow c_1 = y_0 + \frac{a}{b} y_0'$

i.e. $y(t) \to y_0 + \frac{a}{b} y_0'$ as $t \to \infty$.

P184.

2. Sol:

guess the particular sol to be:

$Y(t) = a\cos 2t + b\sin 2t \Rightarrow Y'(t) = -2a\sin 2t + 2b\cos 2t$

$Y''(t) = -4a\cos 2t - 4b\sin 2t$

$\Rightarrow Y''(t) + 2Y'(t) + 5Y(t) = 3\sin 2t$

$\Rightarrow \begin{cases} -4a + 4b + 5a = 0 \\ -4b - 4a + 5b = 3 \end{cases} \Rightarrow \begin{cases} a = -\frac{12}{17} \\ b = \frac{3}{17} \end{cases}$

i.e. $Y(t) = \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$ is a particular sol.

notice that the chara equ: $\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda = -1 \pm 2i$.

i.e. the general sol is:

$$y(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + \frac{3}{17}\sin 2t - \frac{12}{17}\cos 2t$$

8 Sol:

chara equ: $\cancel{2\lambda^2 + 3\lambda + 1 = 0 \Rightarrow \lambda_1 = -1, \ \lambda_2 = -\frac{1}{2}}$

$$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1.$$

guess $Y(t) = t^2 a_0 e^{-t}$

$$Y'(t) = 2a_0 t e^{-t} - a_0 t^2 e^{-t}$$

$$Y''(t) = 2a_0 e^{-t} - 2a_0 t e^{-t} - 2a_0 t e^{-t} + a_0 t^2 e^{-t}$$

$$Y'' + 2Y' + Y = 2a_0 e^{-t} = 2e^{-t} \Rightarrow a_0 = 1.$$

i.e. $Y(t) = t^2 e^{-t}$ is a particular sol.

$\Rightarrow y(t) = C_1 e^{-t} + C_2 t e^{-t} + t^2 e^{-t}$ is the general sol.

9. Sol:

chara equ: $2\lambda^2 + 3\lambda + 1 = 0 \Rightarrow \lambda_1 = -1 \quad \lambda_2 = -\frac{1}{2}$

guess $Y(t) = a\cos t + b\sin t + d_0 + d_1 t + d_2 t^2$

$$Y'(t) = -a\sin t + b\cos t + d_1 + 2d_2 t$$

$$Y''(t) = -a\cos t - b\sin t + 2d_2$$

$$2Y'' + 3Y' + Y = (3b - a)\cos t - (3a + b)\sin t + (4d_2 + 3d_1 + d_0) + (6d_2 + d_1)t + d_2 t^2$$
$$= t^2 + 3\sin t$$

$\Rightarrow a = -\frac{9}{10}, \ b = -\frac{3}{10}, \ d_2 = 1, \ d_1 = -6, \ d_0 = 14.$

i.e. $Y(t) = -\frac{9}{10}\cos t - \frac{3}{10}\sin t + t^2 - 6t + 14$.

the general sol is:

$$y(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} - \frac{9}{10}\cos t - \frac{3}{10}\sin t + t^2 - 6t + 14.$$

18. Sol:

Chara equ: $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 3$

guess $Y(t) = (a_0 + a_1 t)e^{2t}$

$$Y' = 2a_0 e^{2t} + a_1 e^{2t} + 2a_1 t e^{2t}$$

$$Y'' = 4a_0 e^{2t} + 2a_1 e^{2t} + 2a_1 e^{2t} + 4a_1 t e^{2t}$$

$$Y'' - 2Y' - 3Y = -3a_1 t e^{2t} + (2a_1 - 3a_0)e^{2t} = 3te^{2t}$$

$$\Rightarrow a_0 = -\frac{2}{3} \qquad a_1 = -1$$

$$\Rightarrow y(t) = c_1 e^{-t} + c_2 e^{3t} - \frac{2}{3}e^{2t} - te^{2t}.$$

$$y(0) = c_1 + c_2 - \frac{2}{3} = 1$$

$$y'(0) = -c_1 + 3c_2 - \frac{4}{3} - 1 = 0$$

$$\Rightarrow c_1 = \frac{2}{3}. \qquad c_2 = 1$$

i.e. $y(t) = \frac{2}{3}e^{-t} + e^{3t} - \frac{2}{3}e^{2t} - te^{2t}$.

20. Sol:

chara equ: $\lambda^2 + 2\lambda + 5 = 0. \Rightarrow \lambda_1 = -1+2i \quad \lambda_2 = -1-2i$

$$Y(t) = t\{ae^{-t}\cos 2t + be^{-t}\sin 2t\}$$

$\cancel{Y'(t) = ae^{-t}\cos 2t - 2ae^{-t}\sin 2t - be^{-t}\sin 2t + 2be^{-t}\cos 2t}$

$\cancel{Y''(t) = (2b-a)e^{-t}\cos 2t - 2(2b-a)e^{-t}\sin 2t + (2a+b)e^{-t}\sin 2t - 2(2a+b)e^{-t}\cos 2t}$

$\cancel{= -(3a+4b)e^{-t}\cos 2t + (4a-3b)e^{-t}\sin 2t}$.

$$Y'(t) = ae^{-t}\cos 2t + be^{-t}\sin 2t + (2b-a)te^{-t}\cos 2t - (2a+b)te^{-t}\sin 2t.$$

$$Y''(t) = 2(2b-a)e^{-t}\cos 2t - 2(2a+b)e^{-t}\sin 2t$$
$$- (3a+4b)te^{-t}\cos 2t + (4a-3b)te^{-t}\sin 2t.$$

$$Y'' + 2Y' + 5Y = (4b-2a+2a)e^{-t}\cos 2t + (-4a-2b+2b)e^{-t}\sin 2t$$
$$+ (-3a-4b+4b-2a+5a)te^{-t}\cos 2t + (4a-3b-4a-2b+5b)te^{-t}\sin 2t$$
$$= 4e^{-t}\cos 2t$$

$\Rightarrow a = 0,\ b = 1.$

i.e. $Y(t) = te^{-t}\sin 2t.$

$$y(t) = C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t + te^{-t}\sin 2t.$$

$$\begin{cases} y(0) = C_1 = 1 \\ y'(0) = -C_1 + 2C_2 = 0 \end{cases}$$

$\Rightarrow C_1 = 1, \quad C_2 = \frac{1}{2}$

i.e. $y(t) = e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t.$

25. Sol:

$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2.$

$$Y(t) = t^2(a_0 + a_1 t)e^{2t} + (b_0 + b_1 t)\cos 2t + (C_0 + C_1 t)\sin 2t +$$
$$+ d_0 + d_1 t + d_2 t^2.$$

$a_0 = 0 \quad a_1 = \frac{2}{3}, \quad b_0 = \frac{1}{16}, \quad b_1 = \frac{1}{8}, \quad C_0 = -\frac{1}{16}, \quad C_1 = 0, \quad d_0 = \frac{3}{4}, \quad d_1 = 1, \quad d_2 = \frac{1}{2}.$

27. Sol:

$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda_1 = -1, \quad \lambda_2 = -2.$

$$Y(t) = (a_0 t^2 + a_1 t + a_2) e^t \sin 2t + (b_0 t^2 + b_1 t + b_2) e^t \cos 2t + e^{-t}(d \cos t + f \sin t)$$
$$+ g e^t$$

$a_0 = \frac{1}{52}, \quad a_1 = \frac{10}{169} \quad a_2 = -\frac{1233}{35152} \quad b_0 = -\frac{5}{52}, \quad b_1 = \frac{73}{676}, \quad b_2 = -\frac{4105}{35152}$

$d = -\frac{3}{2}, \quad f = \frac{3}{2}, \quad g = \frac{2}{3}.$